ORTHOGONAL POLYNOMIALS ON SEVERAL INTERVALS: ACCUMULATION POINTS OF RECURRENCE COEFFICIENTS AND OF ZEROS

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ABSTRACT. Let $E = \cup_{j=1}^{l}[a_{2j-1}, a_{2j}]$, $a_1 < a_2 < ... < a_{2l}$, $l \geq 2$ and set $\boldsymbol{\omega}(\infty) = (\omega_1(\infty), ..., \omega_{l-1}(\infty))$, where $\omega_j(\infty)$ is the harmonic measure of $[a_{2j-1}, a_{2j}]$ at infinity. Let μ be a measure which is on E absolutely continuous and satisfies Szegő's-condition and has at most a finite number of point measures outside E, and denote by (P_n) and $(\mathcal{Q}_n)$ the orthonormal polynomials and their associated Weyl solutions with respect to $d\mu$, satisfying the recurrence relation $\sqrt{\lambda_{2+n}}y_{1+n} = (x - \alpha_{1+n})y_n - \sqrt{\lambda_{1+n}}y_{-1+n}$. We show that the recurrence coefficients have topologically the same convergence behavior as the sequence $(n\boldsymbol{\omega}(\infty))_{n\in\mathbb{N}}$ modulo 1; More precisely, putting $(\boldsymbol{\alpha}_{1+n}^{l-1}, \boldsymbol{\lambda}_{2+n}^{l-1}) = (\alpha_{[\frac{l-1}{2}]+1+n}, ..., \alpha_{1+n}, ..., \alpha_{-[\frac{l-2}{2}]+1+n}, \lambda_{[\frac{l-2}{2}]+2+n}, ..., \lambda_{2+n}, ..., \lambda_{-[\frac{l-1}{2}]+2+n})$ we prove that $(\boldsymbol{\alpha}_{1+n_\nu}^{l-1}, \boldsymbol{\lambda}_{2+n_\nu}^{l-1})_{\nu\in\mathbb{N}}$ converges if and only if $(n_\nu\boldsymbol{\omega}(\infty))_{\nu\in\mathbb{N}}$ converges modulo 1 and we give an explicit homeomorphism between the sets of accumulation points of $(\boldsymbol{\alpha}_{1+n}^{l-1}, \boldsymbol{\lambda}_{2+n}^{l-1})$ and $(n\boldsymbol{\omega}(\infty))$ modulo 1. As one of the consequences there is a homeomorphism from the so-called gaps $\mathsf{X}_{j=1}^{l-1}$ $([a_{2j}, a_{2j+1}]^+ \cup [a_{2j}, a_{2j+1}]^-)$ on the Riemann surface $y^2 = \prod_{j=1}^{2l}(x - a_j)$ into the set of accumulation points of the sequence $(\boldsymbol{\alpha}_{1+n}^{l-1}, \boldsymbol{\lambda}_{2+n}^{l-1})$ if the harmonic measures $\omega_1(\infty), ..., \omega_{l-1}(\infty)$, 1 are linearly independent over the rational numbers $\mathbb{Q}$. Furthermore it is demonstrated, loosely speaking, that the convergence behavior of the sequence of recurrence coefficients $(\boldsymbol{\alpha}_{1+n}^{l-1}, \boldsymbol{\lambda}_{2+n}^{l-1})$ and of the sequence of zeros of the orthonormal polynomials and Weyl solutions outside the spectrum is topologically the same. The above results are proved by deriving first corresponding statements for the accumulation points of the vector of moments of the diagonal Green's functions, that is, of the sequence $(\int xP_n^2 d\mu, ..., \int x^{l-1}P_n^2 d\mu, \sqrt{\lambda_{2+n}} \int xP_{1+n}P_n d\mu, ..., \sqrt{\lambda_{2+n}} \int x^{l-1}P_{1+n}P_n d\mu)_{n\in\mathbb{N}}$.

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1. Introduction

Let $l \in \mathbb{N}$ and $a_k \in \mathbb{R}$ for $k = 1, ..., 2l, a_1 < a_2 < ... < a_{2l}$. Put

$$E = \bigcup_{k=1}^{l} E_k, \text{ where } E_k = [a_{2k-1}, a_{2k}], \text{ and } H(x) = \prod_{j=1}^{2l} (x - a_j)$$

and henceforth let us choose that branch of $\sqrt{H}$ for which $\sqrt{H(x)} > 0$ for $x \in (a_{2l}, \infty)$. For convenience we set

$$h(x) := \begin{cases} \dfrac{(\sqrt{H})^+(x) - (\sqrt{H})^-(x)}{2i} = (-1)^{l-k}\sqrt{|H(x)|} \text{ for } x \in E_k \\ 0 \quad \text{elsewhere} \end{cases} \tag{1}$$

where, as usual, $f^{\pm}(x)$ denote the limiting values from the upper and lower half plane, respectively. Note that $(\sqrt{H})^-(x) = -(\sqrt{H})^+(x)$ for $x \in E$. By $\phi(z, z_0)$ we denote a so-called complex Green's function for $\overline{\mathbb{C}}\backslash E$ uniquely determined up to a multiplication constant of absolute value one (chosen conveniently below), that is, $\phi(z, z_0)$ is a multiple valued function which is analytic on $\overline{\mathbb{C}}\backslash E$ up to a simple pole at $z = z_0$, has no zeros on $\overline{\mathbb{C}}\backslash E$ and satisfies $|\phi(z, z_0)| \to 1$ for $z \to x \in E$ or in other words $\log |\phi(z, z_0)|$ is the (potential theoretic) Green's function with pole at $z = z_0 \in \overline{\mathbb{C}}\backslash E$, as usual denoted by $g(z, z_0)$. In the case under consideration, as it is known [17, 34], a complex Green's function may be represented as

$$\phi(z) := \phi(z, \infty) = \exp\left(\int_{a_{2l}}^{z} r_\infty(x) \frac{dx}{\sqrt{H(x)}}\right) \tag{2}$$

where r_∞ is the unique monic polynomial of degree $l - 1$ such that

$$\int_{a_{2j}}^{a_{2j+1}} r_\infty(x) \frac{dx}{\sqrt{H(x)}} = 0 \text{ for } j = 0, ..., l - 1; \tag{3}$$

hence

$$r_\infty(x) = \prod_{j=1}^{l-1} (x - c_j), \quad c_j \in (a_{2j}, a_{2j+1}), j = 1, ..., l - 1. \tag{4}$$

Recall that the so-called capacity of E is given by

$$cap(E) = \lim_{z\to\infty} \left| \frac{z}{\phi(z, \infty)} \right| \tag{5}$$

The density of the equilibrium distribution of E, denoted by ρ, becomes

$$\rho(x) = \frac{1}{2\pi i}\left((\frac{\phi'}{\phi})^+(x) - (\frac{\phi'}{\phi})^-(x) \right) = \frac{r_\infty(x)}{\pi h(x)}. \tag{6}$$

By $\omega(z, B, \overline{\mathbb{C}}\backslash E)$ we denote the harmonic measure of $B \subseteq E$ with respect to $\overline{\mathbb{C}}\backslash E$ at z, which is that harmonic and bounded function on $\overline{\mathbb{C}}\backslash E_l$ which

satisfies for $\xi \in E_l$ that $\lim_{z\to\xi}\omega(z, B, \overline{\mathbb{C}}\backslash E_l) = i_B(\xi)$, where i_B denotes the characteristic function of B. For abbreviation we put

$$\omega(z, E_k; \overline{\mathbb{C}}\backslash E) = \omega_k(z).$$

Recall that, $k = 1, ..., l$,

$$\omega_k(\infty) = \int_{a_{2k-1}}^{a_{2k}} \rho(x)dx.$$

In the following we say that the function w is on E from the Szegő-class, written $w \in \mathrm{Sz}(E)$, if

$$\int_E \log\left(w(x)\right)\rho(x)dx > -\infty. \tag{7}$$

In this paper we consider positive measures of the form

$$d\mu(x) = w(x)dx + \sum_{j=1}^{m} \mu_j \delta_{d_j}(x) \tag{8}$$

where $w \in Sz(E)$, the mass points d_j, $j = 1, ..., m$, are from $\mathbb{R}\backslash E$ and mutually disjoint and $\mu_j \in \mathbb{R}^+$. By P_n we denote the polynomial of degree n orthonormal with respect to w i.e.,

$$\int_E P_m(x)P_n(x)d\mu(x) = \delta_{m,n}$$

and the monic orthogonal polynomial are denoted by $p_n(x)$. It is well known that the monic orthogonal polynomials satisfy a three term recurrence relation

$$p_n(x) = (x - \alpha_n)p_{n-1}(x) - \lambda_n p_{n-2}(x) \tag{9}$$

with $p_{-1}(x) := 0$ and $p_0(x) = 1$. For measures of the form (8) it follows, see [20, Cor. 6.1] (in fact more general sets E and measures are considered there), that the recurrence coefficients are almost periodic in the limit, more precisely, that there exist real analytic functions α and λ on the torus $[0, 1]^{l-1}$ and a constant $\boldsymbol{c} \in \mathbb{R}^{l-1}$, depending on the measure μ, such that

$$\alpha_{n+1} = \alpha(n\boldsymbol{\omega}(\infty) - \boldsymbol{c}) + o(1) \quad \text{and} \quad \lambda_{n+2} = \lambda(n\boldsymbol{\omega}(\infty) - \boldsymbol{c}) + o(1), \tag{10}$$

where $\boldsymbol{\omega}(\infty) = (\omega_1(\infty), ..., \omega_{l-1}(\infty))$; for more details see the remark following Proposition 2.3 below. By (10) one gets a first, quite good view into the convergence behavior of the α_n's and λ_n's, but by far not a complete one, since the functions α and λ and their mapping properties are not known. The more it is not clear whether the recurrence coefficients and $(\boldsymbol{\omega}(\infty))_{n\in\mathbb{N}}$ modulo 1 have the same topological convergence behavior; that is, that subsequences converge simultaneously and that there is a homeomorphism between the set of accumulation points. The goal of this paper is to show that, under a proper point of view, there is such a topological equivalence; also between the recurrence coefficients and the zeros of the polynomials and Weyl solutions outside the spectrum. As an immediate consequence

it follows that the problem of convergence of the recurrence coefficients is topologically equivalent to the classical problem in number theory to find diophantic approximations of the harmonic measures $\boldsymbol{\omega}(\infty)$, [22, V. Kapitel], [7, 9].

Instead of continuing to study the accumulation points of the recurrence coefficients with the help of (10), which looks rather hopeless, we may also investigate that ones of the sequences of moments of the diagonal Green's functions, that is, $(\int xP_n^2, \sqrt{\lambda_{n+2}}\int xP_{n+1}P_n, \int x^2P_n^2, \sqrt{\lambda_{n+2}}\int x^2P_{n+1}P_n, ..., \int x^mP_n^2, \sqrt{\lambda_{n+2}}\int x^mP_{n+1}P_n, ...)_{n\in\mathbb{N}}$ since it can be shown (see Proposition 5.2) that there is a unique correspondence between such vectors of moments and the vector of recurrence coefficients $(..., \alpha_{[\frac{m-1}{2}]+1+n}, \lambda_{[\frac{m-2}{2}]+2+n}, ..., \alpha_{1+n}, \lambda_{2+n}, ..., \lambda_{-[\frac{m-1}{2}]+2+n}, \alpha_{-[\frac{m-2}{2}]+1+n}, ...)$. In fact it even suffices to consider the truncated vector sequence $(\int xP_n^2, \sqrt{\lambda_{n+2}}\int xP_{n+1}P_n, ..., \int x^{l-1}P_n^2, \sqrt{\lambda_{n+2}}\int x^{l-1}P_{n+1}P_n)_{n\in\mathbb{N}}$ since, by Corollary 2.5, it carries all information already. We show (Theorem 3.2; concerning the recurrence coefficients see Theorem 5.3) that there is a homeomorphism between the set of accumulation points of the truncated vector sequence of moments (respectively, of recurrence coefficients) and of the sequence $(n\boldsymbol{\omega}(\infty))$ modulo 1 and that convergence holds for the same subsequences. In particular this implies that the set of accumulation points of the truncated vector sequence of moments is homeomorph to a $l-1$ dimensional torus if the harmonic measures are linearly independent over $\mathbb{Q}$, where the homeomorphism is given explicitly even.

Also of special interest is the fact that there is a unique correspondence between the accumulation points of the discussed truncated vector sequence of moments and the accumulation points of zeros of the orthogonal polynomials and Weyl solutions outside the spectrum, see Theorem 4.3.

Next let us represent the weight function w in the form

$$w(x) = \frac{w_0(x)}{\rho(x)} \tag{11}$$

and let $\mathcal{W}_o(z)$ be that function which is analytic on $\tilde{\Omega} := \overline{\mathbb{C}}\backslash E$, has no zeros and poles there, is normalized by $\mathcal{W}(\infty) > 0$ and satisfies the boundary condition

$$w_0(x) = 1/\mathcal{W}_0^+(x)\mathcal{W}_0^-(x). \tag{12}$$

For μ of the form (8) with $w \in \mathrm{Sz}(E)$ it has been shown that P_n/ϕ^n is uniformly asymptotically equivalent to the solution of a certain extremal

problem, more precisely, that on compact subsets of $\Omega := \tilde{\Omega}\backslash\{d_1, ..., d_m\}$

$$\frac{P_n(z)}{\phi^n(z)} \sim \frac{U\mathcal{W}_0(z)\phi^{'}(z)}{\prod_{j=1}^{m}\phi(z;d_j)}\left(\frac{\prod_{j=1}^{l-1}\phi(z;c_j)\prod_{j=1}^{l-1}\phi(z;x_{j,n})^{\delta_{j,n}}\prod_{j=1}^{l-1}(z-x_{j,n})}{r_\infty(z)}\right)^{1/2} \tag{13}$$

where U is a constant and the points $x_{j,n}$ and the $\delta_{j,n} \in \{\pm 1\}$ are uniquely determined by the conditions that for $k = 1, ..., l-1$

$$\begin{aligned}\sum_{j=1}^{l-1}\delta_{j,n}\omega_k(x_{j,n}) = &-(2n-l+1)\omega_k(\infty) - \frac{2}{\pi}\int_E \log\left(\frac{w_0(\xi)}{\rho(\xi)}\right)\frac{\partial\omega_k(\xi)}{\partial n_\xi^+}d\xi \\ &+ 2\sum_{j=1}^{m}\omega_k(d_j) \qquad \text{modulo } 2,\end{aligned} \tag{14}$$

recall that $\omega_k(x) := \omega(x; E_k, \tilde{\Omega})$ is the harmonic measure of E_k with respect to $\tilde{\Omega}$ at the point x and n_ξ^+ is the normal vector at ξ pointing in the upper half plane. It turns out that $x_{j,n} \in [a_{2j}, a_{2j+1}]$ for $j = 1, ..., l-1$ and thus

$$\hat{g}_{(n)}(x; w) := \hat{g}_{(n)}(x) := \prod_{j=1}^{l-1}(x - x_{j,n}) \tag{15}$$

has exactly one zero in each gap $[a_{2j}, a_{2j+1}]$, $j = 1, ..., l-1$. The asymptotic representation (13) is due to Widom [34, Theorem 6.2, p. 168 and pp. 175-176] if there appear no point measures, i.e., if $m = 0$. The case of point measures has been first studied in [23] for two intervals. Asymptotics for more general sets (so-called homogeneous sets) and measures, including the above ones, have been given by P. Yuditskii and the author in [20, 21].

The so-called Weyl solutions or functions of the second kind

$$Q_n(y) = \int_E \frac{p_n(x)}{y-x}d\mu(x), \text{ respectively, } \mathcal{Q}_n(y) = \int_E \frac{P_n(x)}{y-x}d\mu(x) \tag{16}$$

build another basis system of solutions of the recurrence relation (9). In [20, Section 3] also asymptotics for Weyl solutions have been derived, more precisely, using the terminology of this paper, the following uniform asymptotic equivalence on compact subsets of Ω has been shown

$$\mathcal{Q}_n(z) \sim \frac{\prod_{j=1}^{m}\phi(z;d_j)}{U\mathcal{W}_0(z)\phi^{n+1}(z)}\left(\frac{\hat{g}_{(n)}(z)}{r_\infty(z)\prod_{j=1}^{l-1}\phi(z;c_j)\prod_{j=1}^{l-1}\phi(z;x_{j,n})^{\delta_{j,n}}}\right)^{1/2} \tag{17}$$

where $\mathcal{W}_0\mathcal{Q}_n$ is denoted by h_n respectively h in [20].

Finally we mention that other questions about finite gap Jacobi matrices are under investigations by J.S. Christiansen, B. Simon and M. Zinchenko

[3] and that a huge part of the forthcoming book by B. Simon [27], see also [26], is devoted to this topic. The approach is based on automorphic functions similarly as in [20, 21, 28] and is different from that one used here.

2. Asymptotics for principal and secondary diagonal Green's function

The diagonal matrix elements of the resolvent $(\mathcal{J}-z)^{-1}$ are the so-called Green's functions $G(z,n,m)=\langle\delta_n,(\mathcal{J}-z)^{-1}\delta_m\rangle$, where as usual $\mathcal{J}$ denotes the Jacobi matrix associated with (α_j) and (λ_j). More precisely using the orthogonality of P_n, respectively, P_{n+1} one obtains

$$P_n(z)\mathcal{Q}_n(z)=\int\frac{P_n^2(x)}{z-x}d\mu(x)=:G(z,n,n) \tag{18}$$

$$P_n(z)\mathcal{Q}_{n+1}(z)=\int\frac{P_{n+1}(x)P_n(x)}{z-x}d\mu(x)=:G(z,n+1,n) \tag{19}$$

and

$$P_{n+1}(z)\mathcal{Q}_n(z)=\int\frac{P_{n+1}(x)P_n(x)}{z-x}d\mu(x)+\frac{1}{\sqrt{\lambda_{n+2}}}=:G(z,n,n+1). \tag{20}$$

Recall that in the case under consideration the λ_n's are bounded away from zero. Using Schwarz's inequality it follows that the three sequences $(P_n\mathcal{Q}_n),(P_n\mathcal{Q}_{n+1})$ and $(P_{n+1}\mathcal{Q}_n)$ are bounded and analytic on compact subsets of $\mathbb{R}\backslash\mathrm{supp}(\mu)$, hence they are so-called normal families. Combining (13) and (17) we obtain with the help of (2) the following asymptotics for the diagonal Green's function.

Corollary 2.1. *Let μ be given by* (8) *with $w\in\mathrm{Sz}(E)$. Then uniformly on compact subsets of Ω there holds*

$$(P_n\mathcal{Q}_n)(z)=\frac{\hat{g}_{(n)}(z)}{\sqrt{H(z)}}+o(1). \tag{21}$$

For weight functions of the form $v(x)\rho(x)dx$ plus a possible finite number of mass points outside $[a_1,a_{2l}]$, where v is positive and analytic on E, the limit relation (21) has been derived recently in [30] independently from the above asymptotics (13) and (17).

To derive the asymptotic representations of $G(z,n{+}1,n)$ and $G(z,n,n{+}1)$ we will make use of Abel's Theorem and the solvability and uniqueness of the real Jacobi inversion problem. For this reason we have to write Widom's condition (14) in terms of Abelian integrals, which we will do similarly as in [2, 30], see also [1].

Let $\mathfrak{R}$ denote the hyperelliptic Riemann surface of genus $l-1$ defined by $y^2=H$ with branch cuts $[a_1,a_2],[a_2,a_3],\ldots,[a_{2l-1},a_{2l}]$. Points on $\mathfrak{R}$ are written in the form $\mathfrak{z}$ and z denotes the projection of $\mathfrak{z}$ on $\mathbb{C}$ also written $\mathrm{pr}(\mathfrak{z})=z$. Furthermore we also use the notation $\mathrm{pr}(\mathfrak{x})=x$ and $\mathrm{pr}(\mathfrak{y})=y$. $\mathfrak{z}$ and $\mathfrak{z}^*$ denote the points which lie above each other on $\mathfrak{R}$, i.e. $\mathrm{pr}(\mathfrak{z})=\mathrm{pr}(\mathfrak{z}^*)$.

The two sheets of $\mathfrak{R}$ are denoted by $\mathfrak{R}^+$ and $\mathfrak{R}^-$. To indicate that $\mathfrak{z}$ lies on the first respectively second sheet we write $\mathfrak{z}^+$ and $\mathfrak{z}^-$. On $\mathfrak{R}^+$ the branch of $\sqrt{H}$ is chosen for which $\sqrt{H(\mathfrak{x}^+)} > 0$ for $\mathfrak{x}^+ > a_{2l}$.

Furthermore (see e.g. [15, 29]) let the cycles $\{\alpha_j, \beta_j\}_{j=1}^{l-1}$ be the usual canonical homology basis on $\mathfrak{R}$, i.e., the curve α_j lies on the upper sheet $\mathfrak{R}^+$ of $\mathfrak{R}$ and encircles there clockwise the interval E_j and the curve β_j originates at a_{2j} arrives at a_{2l-1} along the upper sheet and turns back to a_{2j} along the lower sheet. Let $\{\varphi_1, \dots, \varphi_{l-1}\}$, where $\varphi_j = \sum_{s=1}^{l-1} e_{j,s} \frac{\mathfrak{z}^s}{\sqrt{H(\mathfrak{z})}} d\mathfrak{z}$, $e_{j,s} \in \mathbb{C}$, be a base of the normalized Abelian differential of the first kind, i.e.,

$$\int_{\alpha_j} \varphi_k = 2\pi i \delta_{jk} \quad \text{and} \quad \int_{\beta_j} \varphi_k = B_{jk} \quad \text{for } j,k = 1, \dots, l-1 \tag{22}$$

where δ_{jk} denotes the Kronecker symbol here. The integrals in (22) are the so-called periods. Note that the $e_{j,n}$'s are real since $\sqrt{H}$ is purely imaginary on E_j and since $\sqrt{H}$ is real on $\mathbb{R} \setminus E$ the symmetric matrix of periods $(B_{j,k})$ is real also.

Abelian differentials of third kind are differentials with simple poles at given points $\mathfrak{x}$ and $\mathfrak{y}$ on $\mathfrak{R}$ with residues $+1$ and -1, respectively, and normalized such that integrals along the $\alpha_\kappa-$cycles vanish for $\kappa = 1, ..., l-1$.

Representing the differential $d \log \phi(z, c)$, $c \in \mathbb{R} \backslash E$, as linear combinations of normalized Abelian differentials of first and third kind (recall that $\phi(\cdot, \mathfrak{c}^+)$, where $c = \mathfrak{c}^+$, has a pole at $\mathfrak{c}^+$ and a zero at $\mathfrak{c}^-$ since the analytic extension of the complex Green's function to the second sheet is given by $\phi(\mathfrak{z}^-, \mathfrak{c}^+) = 1/\phi(\mathfrak{z}^+, \mathfrak{c}^+)$) one obtains by integrating along the α_j and β_j cycles that

$$\int_{\mathfrak{c}^-}^{\mathfrak{c}^+} \varphi_j = \sum_{\kappa=1}^{l-1} \omega_\kappa(c) B_{j\kappa}. \tag{23}$$

Similarly (see [18] for details), since the analytic extension of the harmonic measure

$$w_k = \omega_k + i\omega_k^* \quad \text{on} \quad \mathfrak{R}^+ \quad \text{and} \quad w_k = -\omega_k + i\omega_k^* \quad \text{on} \quad \mathfrak{R}^- \tag{24}$$

is just another basis of Abelian differentials of first kind, φ_j can be represented as linear combination of the w_k's. Integrating along the β_κ cycle, $\kappa \in \{1, ..., l-1\}$, and recalling the fact that the integral along a β_κ-cycle is the difference of values along the α_κ cycle, which is E_κ, we obtain by (24) that

$$\int \varphi_j = -\frac{1}{2} \sum_{\kappa=1}^{l-1} w_\kappa B_{j\kappa}.$$

Hence, using the fact that $\omega_k(z) = 1$ on E_κ, $\kappa \in \{1, ..., l\}$, and thus by the Cauchy-Riemann equations $dw_\kappa(z) = i \frac{\partial \omega_\kappa}{\partial n} ds$, we get

$$\frac{2}{\pi i} \varphi_j^+ = \frac{1}{\pi} \sum_{\kappa=1}^{l-1} \left(\frac{\partial \omega_\kappa}{\partial n^+} ds \right) B_{j\kappa} \tag{25}$$

Notation 2.2. Let $[a_{2j}, a_{2j+1}]^{\pm}$ denote the two copies of $[a_{2j}, a_{2j+1}]$, $j = 1, ..., l-1$ in $\mathfrak{R}^{\pm}$. Note that $[a_{2j}, a_{2j+1}]^{+} \cup [a_{2j}, a_{2j+1}]^{-}$ is a closed loop on $\mathfrak{R}$ and thus $\times_{j=1}^{l-1} ([a_{2j}, a_{2j+1}]^{+} \cup [a_{2j}, a_{2j+1}]^{-})$ is topologically a $l-1$ dimensional torus.

By (23) and (25) Widom's condition (14) becomes,

$$\begin{aligned} \frac{1}{2}\sum_{j=1}^{l-1}\int_{\mathfrak{x}_{j,n}^*}^{\mathfrak{x}_{j,n}} \varphi_k = &- (n - \frac{l-1}{2}) \int_{\infty^-}^{\infty^+} \varphi_k - \frac{i}{\pi}\int_E \varphi_k^+ \log w \\ &+ \sum_{j=1}^{m}\int_{\mathfrak{d}_j^-}^{\mathfrak{d}_j^+} \varphi_k \qquad \text{mod periods} \end{aligned} \tag{26}$$

where $\mathrm{pr}(\mathfrak{x}_{j,n}) = x_{j,n}$ and $\mathfrak{x}_{j,n} \in \mathfrak{R}^{\delta_{j,n}} := \mathfrak{R}^{\pm}$ for $\delta_{j,n} = \pm 1$; recall that $\mathfrak{x}_{j,n}$ and $\mathfrak{x}_{j,n}^*$ lie above each other. Furthermore $\mathrm{pr}(\mathfrak{d}_j^{\pm}) = d_j$.

Next we note that (26) can be considered as so-called Jacobi inversion problem, that is, for given $(\eta_1, ..., \eta_{l-1}) \in \mathrm{Jac}\,\mathfrak{R}$, where $\mathrm{Jac}\,\mathfrak{R}$ denotes the Jacobi variety of $\mathfrak{R}$, (that is the quotient space $\mathbb{C}^{l-1} \backslash (2\pi i \overrightarrow{n} + B\overrightarrow{m})$, $B = (B_{jk})$ the matrix of periods, $\overrightarrow{n}, \overrightarrow{m} \in \mathbb{Z}^{l-1}$) find $\mathfrak{z}_1, ..., \mathfrak{z}_{l-1} \in \mathfrak{R}$ such that

$$\sum_{j=1}^{l-1}\int_{\mathfrak{e}_j}^{\mathfrak{z}_j} \varphi_k = \eta_k \quad \text{mod periods},$$

where $\mathfrak{e}_1, ..., \mathfrak{e}_{l-1}$ are given points on $\mathfrak{R}$. In this paper the η_j's are real always, that is, we deal with the real Jacobi inversion problem. Denoting by $\mathrm{Jac}\,\mathfrak{R}/\mathbb{R} := \mathbb{R}^{l-1}/B\overrightarrow{m}$ the Jacobi variety restricted to reals, the following important uniqueness property of the real Jacobi inversion problem holds (see e.g. [10, 17]): The restricted Abel map

$$\begin{aligned} \mathcal{A} : \times_{j=1}^{l-1}([a_{2j}, a_{2j+1}]^{+} \cup [a_{2j}, a_{2j+1}]^{-}) &\to \mathrm{Jac}\,\mathfrak{R}/\mathbb{R} \\ (\mathfrak{z}_1, ..., \mathfrak{z}_{l-1}) &\mapsto \frac{1}{2}\left(\sum_{j=1}^{l-1}\int_{\mathfrak{z}_j^*}^{\mathfrak{z}_j} \varphi_1, ..., \sum_{j=1}^{l-1}\int_{\mathfrak{z}_j^*}^{\mathfrak{z}_j} \varphi_{l-1}\right) \end{aligned} \tag{27}$$

is a holomorphic bijection.

Now let us show that the solutions $\mathfrak{x}_{1,n}, ..., \mathfrak{x}_{l-1,n}$ of (26) can be described uniquely with the help of two polynomials which is crucial in what follows.

Proposition 2.3. *a) Let $g_{(n)}$, $n \in \mathbb{N}$, be given by* (15)*. There exists a monic polynomial $\hat{f}_{(n+1)}$ of degree l such that*

$$\hat{f}_{(n+1)}^2(x) - H(x) = L_n \hat{g}_{(n+1)}(x)\hat{g}_{(n)}(x) \tag{28}$$

with

$$\hat{f}_{(n+1)}(x_{j,n}) = -\delta_{j,n}\sqrt{H(x_{j,n})} \text{ and } \hat{f}_{(n+1)}(x_{j,n+1}) = -\delta_{j,n+1}\sqrt{H(x_{j,n+1})} \tag{29}$$

where $x_{j,n}, x_{j,n+1}, \delta_{j,n}, \delta_{j,n+1}$ are given by (14) and $L_n \in \mathbb{R}$ is given below. Moreover $\hat{f}_{(n+1)}$ can be represented in the form

$$\hat{f}_{(n+1)}(x) = \left(x + \sum_{j=1}^{l-1} x_{j,n} - c_1 \right) \hat{g}_{(n)}(x) - \sum_{j=1}^{l-1} \frac{\delta_{j,n}\sqrt{H(x_{j,n})}\hat{g}_{(n)}(x)}{\hat{g}'_{(n)}(x_{j,n})(x - x_{j,n})}, \tag{30}$$

where c_1 is given by $H(x) = x^{2l} - 2c_1 x^{2l-1} + \dots$.

Furthermore

$$\frac{(\hat{f}_{(n+1)} \pm \sqrt{H})^2(z)}{L_n \hat{g}_{(n)}(z)\hat{g}_{(n+1)}(z)} = \left(\phi^2(z;\infty) \prod_{j=1}^{l-1} \frac{\phi(z; x_{j,n+1})^{\delta_{j,n+1}}}{\phi(z; x_{j,n})^{\delta_{j,n}}} \right)^{\pm 1} \tag{31}$$

and

$$L_n = 4(\mathrm{cap}(E))^2 \prod_{j=1}^{l-1} \frac{\phi(x_{j,n};\infty)^{\delta_{j,n}}}{\phi(x_{j,n+1};\infty)^{\delta_{j,n+1}}} = 4\lambda_{n+2}(1 + o(1)) \tag{32}$$

b) To each solution $(\mathfrak{x}_{1,n}, \dots, \mathfrak{x}_{l-1,n})$ of (26) there corresponds a unique pair of polynomials $(\hat{g}_{(n)}, \hat{f}_{(1+n)})$ given by (15) and by (30) with $\sqrt{H(\mathfrak{x}_{j,n})} = \delta_{j,n}\sqrt{H(x_{j,n})}$.

Proof. a) Let us write $\mathfrak{x}_{j,n}^{\delta_{j,n}} := \mathfrak{x}_n^{\pm}$ for $\delta_{j,n} = \pm 1$. Considering (26) for n and $n+1$ and substracting both equations we obtain that

$$\sum_{j=1}^{l-1} \int_{\mathfrak{x}_{j,n+1}^{\delta_{j,n+1}}}^{\mathfrak{x}_{j,n+1}^{-\delta_{j,n+1}}} \varphi_k = 2\int_{\infty^-}^{\infty^+} \varphi_k + \sum_{j=1}^{l-1} \int_{\mathfrak{x}_{j,n}^{\delta_{j,n}}}^{\mathfrak{x}_{j,n}^{-\delta_{j,n}}} \varphi_k, \ k = 1, \dots, l-1.$$

Thus it follows from Abel's Theorem that there is a rational function $\mathcal{R}_n$ on the Riemann surface with the following properties

$$\begin{aligned} &\infty^{\pm} \text{ is a double pole (zero)} \\ &\mathfrak{x}_{j,n+1}^{\mp\delta_{j,n+1}} \text{ is a simple zero (pole)} \\ &\mathfrak{x}_{j,n}^{\mp\delta_{j,n+1}} \text{ is a simple pole (zero).} \end{aligned} \tag{33}$$

Thus $\mathcal{R}_n$ can be represented in the form

$$\mathcal{R}_n = \frac{(f_{(n+1)} + \sqrt{H})^2}{L_n \hat{g}_{(n)}\hat{g}_{(n+1)}}, \tag{34}$$

where $f_{(n+1)}$ is a polynomial of degree l and L_n is a constant, since the function at the RHS in (34) satisfies (33) and has no other zeros or poles on $\mathfrak{R}$ as can be checked easily. Since the points $\mathfrak{x}_{j,n+1}^{\pm\delta_{j,n+1}}$ and $\mathfrak{x}_{j,n}^{\pm\delta_{j,n}}$ are real the rational function $\overline{\mathcal{R}_n(\mathfrak{z})}$ has the same properties (33) as $\mathcal{R}_n(\bar{\mathfrak{z}})$, hence $\mathcal{R}_n(\bar{\mathfrak{z}}) = \overline{\mathcal{R}_n(\mathfrak{z})}$ and therefore $f_{(n+1)}$ has real coefficients and $L_n \in \mathbb{R}$. Taking

involution, denoted by $\mathfrak{z}^*$, we obtain by (33)

$$\frac{1}{\mathcal{R}_n(\mathfrak{z})} = \mathcal{R}_n(\mathfrak{z}^*) = \frac{(f_{(n+1)} - \sqrt{H})^2}{L_n \hat{g}_{(n)} \hat{g}_{(n+1)}}, \tag{35}$$

and thus, by multiplying (35) and (34), relation (28)-(29). Concerning (29) note that $\sqrt{H(\mathfrak{x}^\delta)} = \delta\sqrt{H(x)}$.

Next let us note

$$-(\sqrt{H})^-(x) = (\sqrt{H})^+(x) = i(-1)^{l-k}\sqrt{|H(x)|} \text{ for } x \in E_k$$

and therefore by (34) and (28)

$$|\mathcal{R}^\pm(x)| = 1 \text{ for } x \in E.$$

Thus the function

$$F(z) := \frac{\mathcal{R}_n(z)}{\phi^2(z;\infty)} \prod_{j=1}^{l-1} \frac{\phi(z;x_{j,n})^{\delta_{j,n}}}{\phi(z;x_{j,n+1})^{\delta_{j,n+1}}}$$

has neither zeros nor poles on Ω and satisfies $|F^\pm| = 1$ on E. Hence $\log|f(z)|$ is a harmonic bounded function on Ω which has a continuous extension to E and thus $F = 1$, which proves (31).

Considering (31) at $z = \infty$ the first relation in (32) follows. Next denote by $lc(P_n)$ the leading coefficient of P_n. Obviously $\lambda_{n+2} = \int p_{n+1}^2 / \int p_n^2 = (lc(P_n)/lc(P_{n+1}))^2$. Using the fact that $lc(P_n) = \lim_{z\to\infty} P_n(z)/z^n$ it follows by (13), in conjunction with (5), that the last equality in (32) holds.

Relation (30) follows by (29), note that the second term at the RHS of (30) is the unique Lagrange interpolation polynomial which takes on at $x_{j,n}$ the value $-\delta_{j,n}\sqrt{H(x_{j,n})}$, and by equating the first two leading coefficients in (28).

b) follows immediately by a). $\square$

We note in passing that (32), which may be derived from (13) also, is the so-called trace formula for λ_{n+2} and that the other trace formula $-\alpha_{n+1} = \sum_{j=1}^{l-1} x_{j,n} - c_1 + o(1)$ follows immediately by (18), (21) and by equating coefficients of x^{l-1} in (30). By (26) and (27) the $x_{j,n}$'s may be expressed with the help of the Abel map from which representation (10) can be obtained.

For measures from $\mathcal{G}$, $\mathcal{G}$ defined in Corollary 3.4, which are so-called reflectionless measures (note that the set of Jacobi matrices associated with the set of measures $\mathcal{G}$ is the so called isospectral torus), Proposition 2.3 is essentially known [13, 33, 16, 32] apart from the important relation (31). But the way of derivation is exactly the opposite one. First one shows that certain polynomials, which are expressions in Padé approximants, satisfy (28) - (30) - to find the corresponding polynomials in the general case seems to be hopeless, if they exist at all - and then one derives the correspondence with the solutions of (26).

Theorem 2.4. *Let μ be given by (8) with $w \in \mathrm{Sz}(E)$. Then uniformly on compact subsets of Ω there holds*

$$2\sqrt{\lambda_{n+2}}P_n\mathcal{Q}_{n+1} = \frac{\hat{f}_{(n+1)} - \sqrt{H}}{\sqrt{H}} + o(1) \tag{36}$$

$$2\sqrt{\lambda_{n+2}}P_{n+1}\mathcal{Q}_n = \frac{\hat{f}_{(n+1)} + \sqrt{H}}{\sqrt{H}} + o(1) \tag{37}$$

and on compact subsets of $\mathbb{C} \setminus [a_1, a_{2l}]$

$$2\sqrt{\lambda_{n+2}}\frac{P_{n+1}}{P_n} = \frac{\hat{f}_{(n+1)} + \sqrt{H}}{\hat{g}_{(n)}} + o(1) \tag{38}$$

$$\int \frac{1}{z-x}d\mu^{(n+1)}(x) = \sqrt{\lambda_{n+2}}\frac{\mathcal{Q}_{n+1}(z)}{\mathcal{Q}_n(z)} = \frac{\hat{f}_{(n+1)}(z) - \sqrt{H}(z)}{2\hat{g}_{(n)}(z)} + o(1), \tag{39}$$

where $\mu^{(m+1)}$ denotes the measure associated with the $m+1$ forwards shifted recurrence coefficients $(\alpha_{n+m+1})_{n\in\mathbb{N}}$ and $(\lambda_{n+m+2})_{n\in\mathbb{N}}$.

Proof. Concerning relation (36). Plugging in the asymptotic values for P_n and $\mathcal{Q}_{n+1}$ from (13) and (17) and recalling the fact that $\phi'/\phi = r_\infty/\sqrt{H}$ the asymptotic relation follows immediately by (31) and (32).

Analogously (37) is proved. (39) and (38) follow immediately by (36) and Corollary 2.1, respectively (37) and Corollary 2.1. □

Corollary 2.5. *In a neighborhood of $x = 0$ let*

$$\sqrt{H^*(x)} = \sum_{j=0}^{\infty} h_j x^j$$

where $H^*(x) = x^{2l}H(\frac{1}{x})$ is the reciprocal polynomial of H. Then the following limit relations hold for $m \geq l$

$$\lim_n \left(\sum_{j=0}^{m} h_j \int t^{m-j}P_n^2\right) = 0 \tag{40}$$

and for $m \geq l+1$

$$\lim_n \left(h_m + 2\sum_{j=0}^{m-1} h_j \sqrt{\lambda_{2+n}} \int t^{m-1-j}P_{1+n}P_n\right) = 0 \tag{41}$$

Proof. By (21) and (18) it follows that in a neighborhood of $x = 0$

$$\sqrt{H^*(x)}\left(\sum_{j=0}^{\infty}(\int t^j P_n^2)x^j\right) = \hat{g}^*_{(n)}(x) + o(1) \tag{42}$$

Equating coefficients for $m \geq l$ relation (40) follows.

Concerning the second relation we get by (36) that

$$\sqrt{H^*(x)}\left(1+2\sum_{j=0}^{\infty}(\sqrt{\lambda_{2+n}}\int t^j P_{1+n}P_n)x^{j+1}\right)=\hat{f}^*_{(1+n)}(x)+o(1) \tag{43}$$

which proves (41). □

Most likely the limit relations (40) and (41) hold for measures σ whose essential support is E. For the subclass of measures $\mu \in \mathcal{G}$ it can be shown by a different approach, that (40) and (41) hold for $m = l$ and $m = l + 1$, respectively, without limit even, see also [13, 33] where the relations are derived in terms of recurrence coefficients for $l = 2, 3$.

3. Accumulation points of moments of the Green's functions

By series expansion of $1/\sqrt{H(z)}$ at $z = \infty$

$$\sum_{j=0}^{\infty} c_j z^{-(l+j)} = \frac{1}{\sqrt{H(z)}} = \frac{1}{\pi}\int_E \frac{1}{z-t}\frac{dt}{h(t)} \tag{44}$$

for $z \in \mathbb{C}\backslash E$, where the last equality follows by the Sochozki-Plemelj formula. In particular

$$\int_E x^j \frac{dx}{h(x)} = 0 \text{ for } j = 0,...,l-2 \text{ and } c_0 = \int_E x^{l-1}\frac{dx}{h(x)} = 1. \tag{45}$$

Hence the following lemma holds.

Lemma 3.1. *a) Let $A_1,...,A_{l-1} \in \mathbb{R}$ be given. There exists an unique polynomial $P(x) = \sum_{\nu=0}^{l-1} p_\nu x^\nu$ with $p_{l-1} = 1$ such that*

$$\int_E x^j P(x)\frac{dx}{h(x)} = A_j \quad \text{for} \quad j = 1,...,l-1. \tag{46}$$

b) Let $B_1,...,B_{l-1} \in \mathbb{R}$ be given. There exists an unique polynomial $Q(x)$ of degree l with two fixed leading coefficients such that

$$\int_E x^j Q(x)\frac{dx}{h(x)} = B_j \quad \text{for} \quad j = 1,...,l-1. \tag{47}$$

Now we are ready to state our first main result.

Theorem 3.2. *Let μ be given by (8) and let $w \in \mathrm{Sz}(E)$. a) The subsequence of solutions $(\mathfrak{x}_{1,n_\nu},...,\mathfrak{x}_{l-1,n_\nu})_{\nu\in\mathbb{N}}$ of (26) converges if and only if $(\int xP^2_{n_\nu}d\mu,...,\int x^{l-1}P^2_{n_\nu}d\mu, \sqrt{\lambda_{2+n_\nu}}\int xP_{1+n_\nu}P_{n_\nu}d\mu, ..., \sqrt{\lambda_{2+n_\nu}}\int x^{l-1}\, P_{1+n_\nu} P_{n_\nu}d\mu)_{\nu\in\mathbb{N}}$ converges.*

Furthermore, the map τ, given by

$$\begin{aligned}(\mathfrak{y}_1,...,\mathfrak{y}_{l-1}) \mapsto \Bigg(&\int_E xG(x)\frac{dx}{h(x)},...,\int_E x^{l-1}G(x)\frac{dx}{h(x)},\\ &\int_E \frac{xF(x)}{2}\frac{dx}{h(x)},...,\int_E \frac{x^{l-1}F(x)}{2}\frac{dx}{h(x)}\Bigg)\end{aligned} \tag{48}$$

where,
(49)
$$G(x) := \prod_{j=1}^{l-1}(x - y_j), \ \ F(x) := \Big(x + \sum y_j - c_1\Big)\, G(x) - \sum_{j=1}^{l-1} \frac{\delta_j \sqrt{H(y_j)} G(x)}{G'(y_j)(x - y_j)},$$

c_1 *as in* (30) *and* $\delta_j\sqrt{H(y_j)} = \sqrt{H(\mathfrak{y}_j)}$, *is a homeomorphism between the set of accumulation points of the sequence of solutions* $(\mathfrak{x}_{1,n}, ..., \mathfrak{x}_{l-1,n})_{n\in\mathbb{N}}$ *of* (26) *and of the sequence* $(\int xP_n^2 d\mu, ..., \int x^{l-1}P_n^2 d\mu, \sqrt{\lambda_{2+n}} \int x\ P_{1+n}P_n d\mu, ..., \sqrt{\lambda_{2+n}} \int x^{l-1}P_{1+n}P_n d\mu)_{n\in\mathbb{N}}$

b) If $1, \omega_1(\infty), ..., \omega_{l-1}(\infty)$ *are linearly independent over* $\mathbb{Q}$, *then* τ *is a homeomorphism from* $\mathsf{X}_{\mathsf{j=1}}^{\mathsf{l-1}}\ ([a_{2j}, a_{2j+1}]^+ \cup\ [a_{2j}, a_{2j+1}]^-)$ *into the set of accumulation points of* $(\int xP_n^2 d\mu, ..., \int x^{l-1}P_n^2 d\mu, \sqrt{\lambda_{2+n}} \int xP_{1+n}P_n d\mu, ..., \sqrt{\lambda_{2+n}} \int x^{l-1}P_{1+n}P_n d\mu)_{n\in\mathbb{N}}$.

Proof. At the beginning let us observe that the map τ is a continuous one to one map from $\mathsf{X}_{\mathsf{j=1}}^{\mathsf{l-1}}\ ([a_{2j}, a_{2j+1}]^+ \cup [a_{2j}, a_{2j+1}]^-)$ on its range. Indeed, recall first the obvious fact that there is a unique correspondence between a point $\mathfrak{y} \in \mathsf{X}_{\mathsf{j=1}}^{\mathsf{l-1}}\ ([a_{2j}, a_{2j+1}]^+ \cup [a_{2j}, a_{2j+1}]^-)$ and $(y, \delta\sqrt{H(y)}) = (\mathrm{pr}(\mathfrak{y}), \sqrt{H(\mathfrak{y})})$, where $\delta = \pm 1$ if $\mathfrak{y} \in \mathfrak{R}^{\pm}$. Since the polynomial $F(x) - (x + (\sum y_j - c_1)) \cdot G(x)$ is the unique Lagrange interpolation polynomial of degree $l-2$ which takes on at the y_j's the values $-\delta_j\sqrt{H(y_j)}$, it follows that there is a unique correspondence between the points $(\mathfrak{y}_1, ..., \mathfrak{y}_{l-1})$ and the polynomials (G, F), defined in (49). By Lemma 3.1 the observation is proved.

a) Necessity of the first statement. Let $(\mathfrak{x}_{1,n_\nu}, ..., \mathfrak{x}_{l-1,n_\nu})_{\nu\in\mathbb{N}}$ be a sequence of solutions of (26) with limit $(\mathfrak{y}_1, ..., \mathfrak{y}_{l-1})$, that is, $j = 1, ..., l-1$,

$$x_{j,n_\nu} \underset{\nu\to\infty}{\longrightarrow} y_j \text{ and } \delta_{j,n_\nu}\sqrt{H(x_{j,n_\nu})} \underset{\nu\to\infty}{\longrightarrow} \delta_j\sqrt{H(y_j)}.$$

By Proposition 2.3 it follows that there are unique polynomials $\hat{g}_{(n_\nu)}$ and $\hat{f}_{(1+n_\nu)}$ such that uniformly on Ω

$$\hat{g}_{(n_\nu)}(x) \underset{\nu\to\infty}{\longrightarrow} G(x) \text{ and } \hat{f}_{(1+n_\nu)}(x) \underset{\nu\to\infty}{\longrightarrow} F(x), \tag{50}$$

where for the second relation we took (30) into account and that G and F are given by (49).

Next it follows by Corollary 2.1 and (50) that uniformly on compact subsets of Ω

$$\begin{aligned}\lim_\nu P_{n_\nu}\mathcal{Q}_{n_\nu}(z) &= \lim_\nu \int \frac{P_{n_\nu}^2(x)}{z-x} d\mu(x) = \frac{G(z)}{\sqrt{H(z)}} \\ &= \frac{1}{\pi}\int_E \frac{G(x)}{z-x}\frac{dx}{h(x)}\end{aligned} \tag{51}$$

where the last equality follows by the Sochozki-Plemelj's formula using the fact that $G \in \mathbb{P}_{l-1}$. Analogously we obtain by Theorem 2.4 that

$$\begin{aligned}\lim_{\nu} \sqrt{\lambda_{2+n_\nu}} P_{n_\nu} \mathcal{Q}_{1+n_\nu}(z) &= \lim_{\nu} \sqrt{\lambda_{2+n_\nu}} \int \frac{P_{n_\nu}(x) P_{1+n_\nu}(x)}{z-x} d\mu(x) \\ &= \frac{F(z) - \sqrt{H(z)}}{2\sqrt{H(z)}} = \frac{1}{2\pi} \int_E \frac{F(x)}{z-x} \frac{dx}{h(x)}\end{aligned} \tag{52}$$

where the last equality follows by Sochozki-Plemelj again and the fact that $-1 + (F/\sqrt{H})(z) = O(\frac{1}{z})$ as $z \to \infty$, since $f_{(1+n_\nu)}$ has this property by (28). Moreover the first $l-1$ coefficients of the series expansions in (51) and (52) converge which proves the necessity part of the first statement of a).

Furthermore, by (51) and (52) we have shown also, that

$$\lim_{\nu} \left(\int x P_{n_\nu}^2, ..., \int x^{l-1} P_{n_\nu}^2, \int x P_{1+n_\nu} P_{n_\nu}, ..., \int x^{l-1} P_{1+n_\nu} P_{n_\nu} \right)$$

is in the range of the restricted map $\tau_/$ whose domain is the set of accumulation points of the solutions of (26).

Sufficiency of the first statement of a). Suppose that $\lim_{\nu} \left(\int x P_{n_\nu}^2, ..., \right.$ $\int x^{l-1}\ P_{n_\nu}^2, \sqrt{\lambda_{2+n_\nu}}\ \int x P_{1+n_\nu} P_{n_\nu}, ..., \sqrt{\lambda_{2+n_\nu}}\ \int x^{l-1}\ \left. P_{1+n_\nu} P_{n_\nu} \right)$ exists. By (40) and (41) and induction arguments it follows that $\lim_\nu \int x^j P_{n_\nu}^2$ and $\lim_\nu \sqrt{\lambda_{2+n_\nu}} \int x^j P_{1+n_\nu} P_{n_\nu}$ exist for every $j \in \mathbb{N}$ and thus $\lim_\nu \int \frac{P_{n_\nu}^2}{z-x}$ and $\lim_\nu \sqrt{\lambda_{2+n_\nu}} \int \frac{P_{1+n_\nu} P_{n_\nu}}{z-x}$ is uniformly convergent at a neighborhood of $z = \infty$ and thus on compact subsets of $\mathbb{C} \backslash \mathrm{supp}(\mu)$. Corollary 2.1 and Theorem 2.4 imply that the relations (50)-(52) hold, where G, F are by Lemma 3.1 uniquely determined by $\lim_{\nu} \left(\int x P_{n_\nu}^2, ...,\ \int x^{l-1}\ P_{n_\nu}^2, \sqrt{\lambda_{2+n_\nu}} \int x P_{1+n_\nu} P_{n_\nu}, ..., \right.$ $\left. \sqrt{\lambda_{2+n_\nu}}\ \int x^{l-1} P_{1+n_\nu} P_{n_\nu} \right)$.

Now let us take a look at the sequence of solutions of (26) $(\mathfrak{x}_{1,n_\nu}, ...,$ $\mathfrak{x}_{l-1,n_\nu})_{\nu \in \mathbb{N}}$ which can be considered as the sequence $\big((x_{1,n_\nu}, \delta_{1,n_\nu} \sqrt{H(x_{1,n_\nu})}),$ $..., (x_{l-1,n_\nu}, \delta_{l-1,n_\nu} \sqrt{H(x_{l-1,n_\nu})}) \big)_{\nu \in \mathbb{N}}$, where $\delta_{j,n_\nu} = \pm 1$ if $\mathfrak{x}_{j,n_\nu} \in \mathfrak{R}^{\pm}$. Then it follows by Proposition 2.3 if $(\mathfrak{x}_{1,n_\nu}, ..., \mathfrak{x}_{l-1,n_\nu})$ has two different limit points then the uniquely associated sequence of polynomials $(\hat{g}_{(n_\nu)}, \hat{f}_{(1+n_\nu)})$ has two different limit points. But this contradicts (50). Hence there exists $\lim_\nu (\mathfrak{x}_{1,n_\nu}, ..., \mathfrak{x}_{l-1,n_\nu}) = (\mathfrak{y}_1, ..., \mathfrak{y}_{l-1})$ and the first statement of a) is proved.

Furthermore, since (51) - (52) hold, $(\mathfrak{y}_1, ..., \mathfrak{y}_{l-1})$ is mapped by the continuous map $\tau_/$ to $\lim_{\nu} \left(\int x P_{n_\nu}^2, ...,\ \int x^{l-1} P_{n_\nu}^2, \sqrt{\lambda_{2+n_\nu}} \int x P_{1+n_\nu} P_{n_\nu}, ..., \sqrt{\lambda_{2+n_\nu}} \right.$ $\left. \int x^{l-1} P_{1+n_\nu} P_{n_\nu}\ \right)$, that is, $\tau_/$ is an onto map between the two sets of accumulation points and the second statement of a) is proved.

b) We claim that the set of accumulation points of the sequence of solutions of (26) is the set $\mathsf{X}_{\mathsf{j}=1}^{\mathsf{l}-1}\ ([a_{2j}, a_{2j+1}]^+ \cup [a_{2j}, a_{2j+1}]^-)$. Obviously it suffices to show that for given $\boldsymbol{\mathfrak{y}} = (\mathfrak{y}_1, ..., \mathfrak{y}_{l-1}) \in \mathsf{X}_{\mathsf{j}=1}^{\mathsf{l}-1}\ ([a_{2j}, a_{2j+1}]^+ \cup$ $[a_{2j}, a_{2j+1}]^-)$ there exists a sequence $\boldsymbol{\mathfrak{x}}_{n_\nu} := (\mathfrak{x}_{1,n_\nu}, ..., \mathfrak{x}_{l-1,n_\nu})_{\nu \in \mathbb{N}}$ of (26) such that $\lim_\nu \boldsymbol{\mathfrak{x}}_{n_\nu} = \boldsymbol{\mathfrak{y}}$. Since $1, \omega_1(\infty), ..., \omega_{l-1}(\infty)$ is linearly independent it

follows by Kronecker's Theorem [7, 9] that any sequence $(-n\boldsymbol{\omega}(\infty)+\boldsymbol{c})_{n\in\mathbb{N}}$, where $\boldsymbol{c}=(c_1,...,c_{l-1})$ is an arbitrary constant, is, modulo 1, dense in $[0,1]^{l-1}$. Thus by the equivalence of (14) and (26) $(-n\int_{-\infty}^{\infty}\boldsymbol{\varphi}+\tilde{\boldsymbol{c}})_{n\in\mathbb{N}}$ is modulo periods B_{jk}, dense in Jac $\mathfrak{R}/\mathbb{R}$ where the constant $\tilde{\boldsymbol{c}}=(\tilde{c}_1,...,\tilde{c}_{l-1})$ is given by that part of the RHS of (26) which does not depend on n and where we have put $\boldsymbol{\varphi}=(\varphi_1,...,\varphi_{l-1}),$. Hence there exists a subsequence (n_ν) of natural numbers such that

$$\left(-n_\nu\int_{-\infty}^{\infty}\boldsymbol{\varphi}+\tilde{\boldsymbol{c}}\right)_{\nu\in\mathbb{N}}\to\mathcal{A}(\mathfrak{y})\qquad\text{modulo periods }B_{jk},\tag{53}$$

where $\mathcal{A}$ is the Abel map from (27). On the other hand to the sequence $(P_{n_\nu})_{\nu\in\mathbb{N}}$ of orthonormal polynomials there exist points $\mathfrak{x}_{n_\nu}=(\mathfrak{x}_{1,n_\nu},...,\mathfrak{x}_{l-1,n_\nu})$ such that (26) holds, that is, that

$$\mathcal{A}(\mathfrak{x}_{n_\nu})=-n_\nu\int_{-\infty}^{+\infty}\boldsymbol{\varphi}+\tilde{\boldsymbol{c}}\qquad\text{modulo periods }B_{jk},$$

By (53) and the bijectivity of $\mathcal{A}$ the assertion follows. □

Remark 3.3. If one wants to get rid of the Riemann-surface the homeomorphism from Theorem 3.2 a) may be written also as the map from $A:=\{((y_1,\delta_1),...,(y_{l-1},\delta_{l-1})):y_j\in[a_{2j},a_{2j+1}],\ \delta_j\in\{-1,1\}$ if $y_j\in(a_{2j},a_{2j+1})$ and $\delta_j=0$ if $y_j\in\{a_{2j},a_{2j+1}\}$ for $j=1,...,l-1\}$ into the set of accumulation points of the sequence $(\int xP_n^2,...,\int x^{l-1}P_n^2,\sqrt{\lambda_{2+n}}\int x^{l-1}P_{1+n}P_n,...,\sqrt{\lambda_{2+n}}\ \int x^{l-1}P_{1+n}\ P_n)_{n\in\mathbb{N}}$, where $((y_1,\delta_1),...,(y_{l-1},\delta_{l-1}))\mapsto(\int x\ G(x)\ \frac{dx}{h(x)},...,\int x^{l-1}\ G(x)\ \frac{dx}{h(x)},\int\frac{xF(x)}{2}\frac{dx}{h(x)},...,\int\frac{x^{l-1}F(x)}{2}\frac{dx}{h(x)})$.

Corollary 3.4. *Suppose that the harmonic measures* $1,\omega_1(\infty),...,\omega_{l-1}(\infty)$ *are linearly independent over* $\mathbb{Q}$. *Then the following statements hold:*

a) The set of limit points of the associated measures $\{\mu^{(n)}\}_{n\in\mathbb{N}}$, *see* (39), *with respect to weak convergence is the set of measures*

$$\begin{aligned}\mathcal{G}:=\{&\frac{h(t)}{2\pi\prod_{j=1}^{l-1}(t-y_j)}dt+\sum_{j=1}^{l-1}\frac{1-\delta_j}{2}\frac{\sqrt{H(y_j)}}{\frac{d}{dt}(\prod_{j=1}^{l-1}(t-y_j))_{t=y_j}}\delta(t-y_j):\\&y_j\in[a_{2j},a_{2j+1}],\delta_j\in\{\pm1\}\text{ for }j=1,...,l-1\}\,.\end{aligned}$$

b) Let $\mathfrak{s}$ *be the map associated with coefficient stripping, i.e.* $\mathfrak{s}(\mu)=\mu^{(1)}$. *Then* $\mathfrak{s}(\mathcal{G})\subseteq\mathcal{G}$ *and for every* $\mu\in\mathcal{G}$ *the orbit under composition* $\{\mathfrak{s}^n(\mu):n\in\mathbb{N}\}$ *is dense in* $\mathcal{G}$ *with respect to weak convergence.*

Proof. a) By (39) on compact subsets of $\mathbb{C}\setminus[a_1,a_{2l}]$

$$\int\frac{1}{z-t}d\mu^{(1+n)}=\sqrt{\lambda_{2+n}}\frac{\mathcal{Q}_{1+n}(z)}{\mathcal{Q}_n(z)}=\frac{\hat{f}_{(1+n)}(z)-\sqrt{H(z)}}{2\hat{g}_{(n)}(z)}+o(1)$$

In the proof of Theorem 3.2 we have shown that for each $\Big((y_1, \delta_1\sqrt{H(y_1)}, ...,$ $(y_{l-1}, \delta_{l-1}\delta_1\sqrt{H(y_{l-1})})\Big)$ there is a sequence (n_ν) such that

$$\hat{g}_{(n_\nu)} \underset{\nu\to\infty}{\longrightarrow} G \text{ and } \hat{f}_{(1+n_\nu)} \underset{\nu\to\infty}{\longrightarrow} F,$$

where G and F are given by (49). Since

$$\begin{aligned}\frac{\hat{f}_{(1+n)}(z) - \sqrt{H(z)}}{2\hat{g}_{(n)}(z)} &= \frac{1}{2\pi}\int_E \frac{1}{z-t}\frac{h(t)}{\hat{g}_{(n)}(t)}dt \\ &\quad + \sum \frac{(1-\delta_{j,n})}{2}\frac{\sqrt{H(x_{j,n})}}{\hat{g}'_{(n)}(x_{j,n})}\delta(z - x_{j,n})\end{aligned}$$

the assertion follows.

b) The invariance with respect to coefficient stripping has been proved in [16, Theorem 5]. Applying part a) to $\mu \in \mathcal{G}$ the assertion follows. □

The set of Jacobi matrices associated with the set of measures $\mathcal{G}$ is called the isospectral torus nowadays.

4. Accumulation points of zeros outside the support

It is well known that polynomials orthogonal with respect to a measure σ may have zeros in the convex hull of $\mathrm{supp}(\sigma)$, that is, in the case under consideration in the gaps $[a_{2j}, a_{2j+1}], j \in \{1, ..., l-1\}$. The same holds for the Weyl solutions $\mathcal{Q}_n$.

Notation 4.1. Let (n_j) be a strictly monotone increasing subsequence of the natural numbers and let (f_{n_j}) be a sequence of functions. We say that a point y is an accumulation point of zeros of (f_{n_j}) if there exists a sequence of points (y_j) such that $f_{n_j}(y_j) = 0$ and $\lim_j y_j = y$. As usual, y is called a limit point of zeros of (y_j) if $U_\varepsilon(y)\setminus\{y\}$, $\varepsilon > 0$, contains an infinite number of y_j's.

Lemma 4.2. *Let σ be a positive measure with* $\mathrm{supp}(\sigma)$ *bounded and suppose that $0 < const \le \lambda_n$ for $n \ge n_0$. Then the following pairs of sequences have no common accumulation point of zeros on $\mathbb{R}\backslash\mathrm{supp}(\sigma)$: (P_{n_j}) and (P_{1+n_j}), $(\mathcal{Q}_{n_j})$ and $(\mathcal{Q}_{1+n_j})$, and (P_{n_j}) and $(\mathcal{Q}_{n_j})$.*

Proof. Since the three sequences $(P_n\mathcal{Q}_n), (P_n\mathcal{Q}_{n+1})$ and $(P_{n+1}\mathcal{Q}_n)$ are normal families on $\mathbb{R}\backslash\mathrm{supp}(\mu)$, see (18)-(20), they are equicontinuous. Thus the assertion follows by the well known relation

$$P_n\mathcal{Q}_{n+1} - \mathcal{Q}_nP_{n+1} = -1. \qquad \square$$

By the way that (P_{n_j}) and (P_{1+n_j}) have no common accumulation point if $y \notin \mathrm{supp}(\sigma)$ follows also from [5]. Thus y, $y \notin \mathrm{supp}\ \sigma$, is an accumulation point of zeros of (P_{n_ν}) $((\mathcal{Q}_{n_\nu}))$ if and only if y is a common accumulation

point of zeros of $(P_{n_\nu}\mathcal{Q}_{n_\nu})$ and $(P_{n_\nu}\mathcal{Q}_{1+n_\nu})$ (of $(P_{n_\nu}\mathcal{Q}_{n_\nu})$ and $(P_{1+n_\nu}\mathcal{Q}_{n_\nu})$). In the following it will be convenient to use this way of expression.

Theorem 4.3. *Let μ be given by* (8) *with $w \in \mathrm{Sz}(E)$. Then*

(54)
$$\lim_{\nu}\Big(\int xP_{n_\nu}^2 d\mu, ..., \int x^{l-1}P_{n_\nu}^2 d\mu, \sqrt{\lambda_{2+n_\nu}}\int xP_{1+n_\nu}P_{n_\nu}d\mu, ..., \sqrt{\lambda_{2+n_\nu}}\int x^{l-1}P_{1+n_\nu}P_{n_\nu}d\mu\Big)$$
$$= \Big(\int_E xG(x)\frac{dx}{h(x)}, ..., \int_E x^{l-1}G(x)\frac{dx}{h(x)}, \int_E \frac{xF(x)}{2}\frac{dx}{h(x)}, ..., \int_E \frac{x^{l-1}F(x)}{2}\frac{dx}{h(x)}\Big),$$

where G and F are defined in (49), $((y_1,\delta_1), ..., (y_{l-1},\delta_{l-1})) \in \times_{j=1}^{l-1}$ $(((a_{2j},a_{2j+1})\backslash supp(\mu)) \times (\{\pm 1\}))$, *if and only if for $j = 1, ..., l-1$ the point y_j, $y_j \in (a_{2j},a_{2j+1})\backslash supp(\mu)$, is a common accumulation point of zeros of $(P_{n_\nu}Q_{n_\nu})$ and $(P_{(1+\delta_j)/2+n_\nu}\ Q_{(1-\delta_j)/2+n_\nu})$, $\delta_j \in \{\pm 1\}$.*

Proof. Necessity. In the proof of the sufficiency part of Theorem 3.2a) we have shown that relation (54) implies with the help of Corollary 2.5 that the relations (50) hold. Moreover by (30) the δ_{j,n_ν}'s from (29) satisfy $\delta_{j,n_\nu} \underset{\nu\to\infty}{\longrightarrow} \delta_j$, $j = 1, ..., l-1$, where $\delta_j \in \{-1, 1\}$, since every zero of G lies in the interior of the gaps. Taking a look at the three limit relations (21), (36) and (37) in conjunction with (29) the assertion is proved using Hurwitz's Theorem [8] about the zeros of uniform convergent sequences of analytic functions.

Sufficiency. First let us note that for any subsequence (n_κ) of (n_ν) for which the limit in (21), (36) and (37) exists the limit functions $\lim_\kappa \hat{g}_{(n_\kappa)} = G$ and $\lim_\kappa \hat{f}_{(1+n_\kappa)} = F$ are monic polynomials of degree $l-1$ and l, respectively, which satisfy, using the assumption and Hurwitz Theorem, $G(y_j) = 0$ and $F(y_j) = \delta_j\sqrt{H(y_j)}$ for $j = 1, ..., l-1$. Thus G and F, recall (30), are unique; in other words the limit of all three sequences $(P_{n_\nu}\mathcal{Q}_{n_\nu})$, $(P_{n_\nu}\mathcal{Q}_{1+n_\nu})$ and $(P_{1+n_\nu}\mathcal{Q}_{n_\nu}))$ exists uniformly on Ω and is given by G and F which proves by (18)-(20), taking into consideration the last relation from (51) and (52) respectively, the sufficiency part. □

Combining Theorem 4.3 and Theorem 3.2 b) we obtain immediately (for a wider class of measures) a result of the author [17, Theorem 3.9] concerning the denseness of zeros of (P_n) in the gaps. For absolutely continuous, smooth measures the existence of a sequence (n_ν) such that (P_{n_ν}) has no zeros in the gaps was shown first in [30].

5. Consequences for the recurrence coefficients

First let us demonstrate how to express Theorem 4.3 in terms of limit relations of the recurrence coefficients and of the accumulation points of zeros of (P_n) and $(\mathcal{Q}_n)$. It is well known that $\int x^jP_n^2$ and $\sqrt{\lambda_{n+2}}\int x^jP_{n+1}$

P_n, $j \in \mathbb{N}$, can be expressed in terms of the recurrence coefficients using the recurrence relation of the P_n's, e.g.

$$\int xP_n^2 = \alpha_{n+1}, \int x^2 P_n^2 = \lambda_{n+2} + \alpha_{n+1}^2 + \lambda_{n+1}, ...$$

and

$$\sqrt{\lambda_{n+2}} \int xP_{n+1}P_n = \lambda_{n+2}, \sqrt{\lambda_{n+2}} \int x^2 P_{n+1}P_n = \lambda_{n+2}(\alpha_{n+2} + \alpha_{n+1}), ...$$

for a closed formula see [14]. Solving the system of equations (recall Lemma 3.1)

$$\int_E x^j G(x) \frac{dx}{h(x)} = \lim_{\nu} \int x^j P_{n_\nu}^2 \quad j = 0, ..., l-1, \tag{55}$$

respectively,

$$\int_E x^j F(x) \frac{dx}{h(x)} = \lim_{\nu} 2\sqrt{\lambda_{2+n_\nu}} \int x^j P_{1+n_\nu} P_{n_\nu} \quad j = 0, ..., l-1 \tag{56}$$

with the help of (44) and (45) we obtain explicit expressions for the coefficients of $G(x)$, respectively, $F(x)$ in terms of the coefficients c_j of the series expansion of $1/\sqrt{H(z)}$, see (44), and of the accumulation points of the recurrence coefficients. This enables us to write condition (54) of Theorem 4.3 in terms of accumulation points of recurrence coefficients and of zeros only. Let us demonstrate this for the case of two and three intervals.

Corollary 5.1. *Let $E = [a_1, a_2] \cup [a_3, a_4]$, $d\mu = w(x)dx$ with $w \in \mathrm{Sz}(E)$, and let $\delta \in \{\pm 1\}$. Then $y, y \in (a_2, a_3)$, is a common accumulation point of $(P_{n_\nu} \mathcal{Q}_{n_\nu})$ and $(P_{(1+\delta)/2+n_\nu} \mathcal{Q}_{(1-\delta)/2+n_\nu})$ if and only if*

$$\lim_{\nu} \alpha_{1+n_\nu} = -y + c_1 \text{ and } \lim_{\nu} \lambda_{2+n_\nu} = y(c_1 - y) + c_2 - c_1^2 - \delta\sqrt{H(y)} \tag{57}$$

Furthermore, for every $(y, \delta) \in (a_2, a_3) \times \{-1, 1\}$ there exists a subsequence (n_ν) of the natural numbers such that (57) holds, if E is not the inverse image under a polynomial map.

If E is the inverse image under a polynomial map of degree N then for each $b = 0, ..., N-1$, either $(P_{\nu N+b})_{n\in\mathbb{N}}$ or $(\mathcal{Q}_{\nu N+b})_{n\in\mathbb{N}}$ has an accumulation point of zeros at the zero y of the $b-$th associated polynomial $p_{N-1}^{(b)}$ which lies in (a_2, a_3). The limits of $(\alpha_{1+\nu N+b})$ and $(\lambda_{2+\nu N+b})$ are given by (57), where $\delta = \mp 1$ if y is an accumulation point of zeros of $(P_{\nu N+b})$, respectively, of $(\mathcal{Q}_{\nu N+b})$.

Proof. Solving the system (55) and (56) of equations we obtain that

$$G(x) = x + \lim_{\nu} \alpha_{1+n_\nu} - c_1 \text{ and } F(x) = x^2 - c_1 x + 2\lim_{\nu} \lambda_{2+n_\nu} + c_1^2 - c_2 \tag{58}$$

By (49) and the statements of Theorem 4.3 the assertion follows, if E is not the inverse image.

In the case when E is the inverse image under a polynomial see [19]. □

In the case when $E=[a_1,a_2]\cup[a_3,a_4]\cup[a_5,a_6]$ the solution of the system of equations (55) and (56) yields

(59)
$$\begin{aligned}G(x) &= x^2+(\tilde\alpha_{1+n_\nu}-c_1)x+\tilde\lambda_{2+n_\nu}+\tilde\lambda_{1+n_\nu}+\tilde\alpha^2_{1+n_\nu}-\tilde\alpha_{1+n_\nu}c_1+c_1^2-c_2\\ F(x) &= x^3-c_1x^2+(2\tilde\lambda_{2+n_\nu}+c_1^2-c_2)x+2\tilde\lambda_{2+n_\nu}(\tilde\alpha_{2+n_\nu}+\tilde\alpha_{1+n_\nu}-c_1)\\ &\quad -(c_1^3-2c_1c_2+c_3)\end{aligned}$$

where tilde denotes the limits, that is, $\tilde\alpha_{1+n_\nu}:=\lim_\nu \alpha_{1+n_\nu},\ldots$. Equating coefficients with the polynomials in (49) gives easily the condition on the limits of the recurrence coefficients such that y_j, $y_j\in(a_{2j},a_{2j+1})$, $j=1,2$, are common accumulation points of zeros of $(P_{n_\nu}\mathcal{Q}_{n_\nu})$ and $(P_{(1+\delta_j)/2+n_\nu}\mathcal{Q}_{(1+\delta_j)/2+n_\nu})$, $j=1,2$, for given $\delta_j\in\{\pm1\}$, $j=1,2$.

To obtain a complete picture of the behaviour of the accumulation points of the recurrence coefficients let us first show that there is a unique correspondence to that ones of the moments of the Green's functions.

Proposition 5.2. *a) Let $m\in\mathbb{N}$, $(\mathbf{x}_m,\mathbf{y}_m):=(x_{[\frac{m}{2}]+1},\ldots,x_1,\ldots,x_{-[\frac{m-1}{2}]+1},$ $y_{[\frac{m-1}{2}]+2},\ldots,y_2,\ldots,y_{-[\frac{m}{2}]+2})$, where $x_j,y_j\in\mathbb{R}$, and put for $k,j\in\mathbb{N}_0$, $j\geq k$,*

$$I_{j,k}:=I_{j,k}(\mathbf{x}_m,\mathbf{y}_m) = \sum_{\substack{-1\le k_i\le 1\\ i=1,2,\ldots,j,\\ \sum_{i=1}^j k_i=k}} z_{0,k_1}z_{k_1,k_1+k_2}\ldots z_{k_1+\ldots+k_{j-1},k_1+\ldots+k_j} \tag{60}$$

where

$$z_{k,j}=\begin{cases}\sqrt{y_{j+1}} & k=j-1\\ x_{j+1} & k=j\\ \sqrt{y_{j+2}} & k=j+1\end{cases}$$

Then

$$\begin{aligned}&\mathcal{I}_m:\mathbb{R}^m\times\mathbb{R}^m_+\to\mathbb{R}^{2m}\\ &\quad(\mathbf{x}_m,\mathbf{y}_m)\longmapsto(I_{1,0},\sqrt{y_2}I_{1,1},I_{2,0},\sqrt{y_2}I_{2,1},\ldots,I_{m,0},\sqrt{y_2}I_{m,1})\end{aligned}$$

is a continuous one to one map.

b) Let σ be a positive measure and suppose that $(\alpha_{1+n}(d\sigma),\lambda_{2+n}(d\sigma))$ is bounded and $(\lambda_{2+n}(d\sigma))_{n\in\mathbb{N}}$ is bounded away from zero. Let $m\in\mathbb{N}$ be fixed, and put $(\boldsymbol{\alpha}^m_{1+n}(d\sigma),\boldsymbol{\lambda}^m_{2+n}(d\sigma))_{n\in\mathbb{N}}:=(\alpha_{[\frac{m}{2}]+1+n}(d\sigma),\ldots,\alpha_{1+n}(d\sigma),\ldots,$ $\alpha_{-[\frac{m-1}{2}]+1+n}(d\sigma),\lambda_{[\frac{m-1}{2}]+2+n}(d\sigma),\ldots,\lambda_{2+n}(d\sigma),\ldots,\lambda_{-[\frac{m}{2}]+2+n}(d\sigma))_{n\in\mathbb{N}}$. Then for $n>m$

$$\begin{aligned}\mathcal{I}_m((\boldsymbol{\alpha}^m_{1+n}(d\sigma),\boldsymbol{\lambda}^m_{2+n}(d\sigma)) &= (\int xP_n^2d\sigma,\sqrt{\lambda_{2+n}(d\sigma)}\int xP_nP_{1+n}d\sigma,\\ &\qquad\ldots,\int x^mP_n^2d\sigma,\sqrt{\lambda_{2+n}(d\sigma)}\int x^mP_nP_{1+n}d\sigma)\end{aligned} \tag{61}$$

and $\mathcal{I}_m$ is a homeomorphism between the set of accumulation points of the two sequences.

Proof. a) First let us note that by (60) $I_{j,k}$, $k \in \{0,1\}$, $j \in \{1,...,m\}$, depends on the variables $(\mathbf{x}_j, \mathbf{y}_j)$ only and that

(62)
$$I_{m,0}(\mathbf{x}_m, \mathbf{y}_m) = \begin{cases} y_{-\frac{m}{2}+2} \prod_{i=-\frac{m}{2}+3}^{1} y_i + e_1(\mathbf{x}_{m-1}, \mathbf{y}_{m-1}) & \text{if } m \text{ is even} \\ x_{-(\frac{m-1}{2})+1} \prod_{i=-(\frac{m-1}{2})+2}^{1} y_i + e_2(\mathbf{x}_{m-1}, \mathbf{y}_{m-1}) & \text{if } m \text{ is odd} \end{cases}$$

since in (60) the lowest possible indices i of x_i, y_i appear only for the choice $(k_1, ..., k_m) = (-1, ..., -1, 1, ..., 1)$, where ∓ 1 appears $m/2-$times, respectively for $(-1, ..., -1, 0, 1, ..., 1)$ where ∓ 1 appears $(m-1)/2$ times. By the way, the highest possible indices are obtained by reversing the order, that is, for the choice $(1, ..., 1, -1, ..., -1)$, respectively, $(1, ..., 1, 0, -1, ..., -1)$ which is needed later.

Furthermore for $m > 1$

$$\frac{I_{m,1}(\mathbf{x}_m, \mathbf{y}_m)}{\sqrt{y_2}} = \begin{cases} x_{\frac{m}{2}+1} \prod_{i=3}^{\frac{m}{2}+1} y_i \; + \; e_3(\mathbf{x}_{m-1}, \mathbf{y}_{m-1}) & \text{if } m \text{ is even} \\ y_{\frac{m-1}{2}+2} \prod_{i=3}^{\frac{m-1}{2}+1} y_i \; + e_4(\mathbf{x}_{m-1}, \mathbf{y}_{m-1}) & \text{if } m \text{ is odd} \end{cases} \tag{63}$$

since in (60) the highest possible indices i of x_i, y_i appear only for the choice $(k_1, ..., k_m) = (1, ..., 1, 0, -1, ..., -1)$, where $+1$ appears $m/2$ and -1 appears $(m-2)/2$ times, respectively, for $(1, ..., 1, -1, ..., -1)$ where ± 1 appears $(m \pm 1)/2$ times. Now we are able to prove the assertion, that is,

$$\mathcal{I}_m(\mathbf{x}_m, \mathbf{y}_m) = \mathcal{I}_m(\tilde{\mathbf{x}}_m, \tilde{\mathbf{y}}_m) \text{ implies } (\mathbf{x}_m, \mathbf{y}_m) = (\tilde{\mathbf{x}}_m, \tilde{\mathbf{y}}_m) \tag{64}$$

by induction arguments with respect to m. Since $I_{j,k}$, $k \in \{0,1\}$, depends for $j = 0, ..., m-1$ on $(\tilde{\mathbf{x}}_{m-1}, \tilde{\mathbf{y}}_{m-1})$ only it follows by the induction hypothesis that

$$(\mathbf{x}_{m-1}, \mathbf{y}_{m-1}) = (\tilde{\mathbf{x}}_{m-1}, \tilde{\mathbf{y}}_{m-1}) \tag{65}$$

Thus it remains to be shown that

$$x_{\pm[\frac{m}{2}]+1} = \tilde{x}_{\pm[\frac{m}{2}]+1} \text{ and } y_{\mp[\frac{m}{2}]+2} = \tilde{y}_{\mp[\frac{m}{2}]+2} \tag{66}$$

if m is even, respectively odd. Since by (64)

$$I_{m,0}(\tilde{\mathbf{x}}_m, \tilde{\mathbf{y}}_m) = I_{m,0}(\mathbf{x}_m, \mathbf{y}_m) \text{ and } I_{m,1}(\tilde{\mathbf{x}}_m, \tilde{\mathbf{y}}_m) = I_{m,1}(\mathbf{x}_m, \mathbf{y}_m)$$

it follows by (62) and (63) in conjunction with (65) that (66) holds and thus part a) is proved.

b) In [14] it has been shown that, $n > j$,

$$I_{j,k}(\boldsymbol{\alpha}^m_{1+n}(d\sigma), \boldsymbol{\lambda}^m_{2+n}(d\sigma)) = \int x^j P_n P_{k+n} d\sigma \tag{67}$$

which gives (61). By the assumptions on the recurrence coefficients it follows that the set of accumulation points is a compact set contained in $\mathbb{R}^m \times \mathbb{R}^m_+$ and thus its image under $\mathcal{I}_m$ is a compact set contained in the range of $\mathcal{I}_m$, which is by (67) and continuity of $\mathcal{I}_m$ equal to the set of accumulation points of $(\int xP_n^2 d\sigma,\ \sqrt{\lambda_{2+n}}\int xP_nP_{1+n}, ...,\ \int x^mP_n^2 d\sigma,$ $\sqrt{\lambda_{2+n}}\int x^mP_nP_{1+n})_{n\in\mathbb{N}}$. □

We point out that the starting point of Proposition 5.2a) was formula (67) due to Nevai [14]. Combining Theorem 3.2 a) and Proposition 5.2 b) we obtain

Theorem 5.3. *Let μ be given by (8) with $w \in Sz(E)$ and put $(\boldsymbol{\alpha}^{l-1}_{1+n}(d\mu)$, $\boldsymbol{\lambda}^{l-1}_{2+n}(d\mu))_{n\in\mathbb{N}} := (\alpha_{[\frac{l-1}{2}]+1+n}(d\mu), ..., \alpha_{1+n}(d\mu), ..., \alpha_{-[\frac{l-2}{2}]+1+n}(d\mu)$, $\lambda_{[\frac{l-2}{2}]+2+n}(d\mu), ..., \lambda_{2+n}(d\mu), ..., \lambda_{-[\frac{l-1}{2}]+2+n}(d\mu))_{n\in\mathbb{N}}$.*

a) $(\boldsymbol{\alpha}^{l-1}_{1+n_\nu}(d\mu), \boldsymbol{\lambda}^{l-1}_{2+n_\nu}(d\mu))_{\nu\in\mathbb{N}}$ converges if and only if

(68)
$$\left((n_\nu - (l-1)/2)\,\omega_k(\infty) + \frac{1}{\pi}\int_E \log\left(w(\xi)\right)\frac{\partial\omega_k(\xi)}{\partial n^+_\xi}d\xi - \sum_{j=1}^m \omega_k(d_j)\right)_{\nu\in\mathbb{N}}$$

converges modulo 1 for $k = 1, ..., l-1$. Furthermore $\mathcal{T}\circ\mathcal{A}\circ\tau^{-1}\circ\mathcal{I}_{l-1}$ is a homeomorphism between the sets of accumulation points, where $\mathcal{A}$ is the Abel map from (27), τ is given in Theorem 3.2 and $\mathcal{T}$ is the map between Jac $\mathfrak{R}/\mathbb{R}$ *and the real torus $[0,1]^{l-1}$.*

b) $\tau^{-1}\circ\mathcal{I}_{l-1}$ is a homeomorphism from the set of accumulation points of $(\boldsymbol{\alpha}^{l-1}_{1+n}(d\mu), \boldsymbol{\lambda}^{l-1}_{2+n}(d\mu))_{n\in\mathbb{N}}$ into the torus $\times_{j=1}^{l-1}([a_{2j}, a_{2j+1}]^+ \cup [a_{2j}, a_{2j+1}]^-)$, if $1, \omega_1(\infty), ..., \omega_{l-1}(\infty)$ are linearly independent over $\mathbb{Q}$.

Proof. a) By Proposition 5.2b) and Theorem 3.2 a) the sequence $(\boldsymbol{\alpha}^{l-1}_{1+n_\nu}, \boldsymbol{\lambda}^{l-1}_{2+n_\nu})_{\nu\in\mathbb{N}}$ converges if and only if $(\mathfrak{x}_{1,n_\nu}, ..., \mathfrak{x}_{l-1,n_\nu})_{\nu\in\mathbb{N}}$ converges, where $(\mathfrak{x}_{1,n_\nu}, ..., \mathfrak{x}_{l-1,n_\nu})$ is given by (26). Next let us recall that, by the form of Jac $\mathfrak{R}/\mathbb{R}$ and the bijectivity of the Abel map $\mathcal{A}$, for every $(\mathfrak{x}_1, ..., \mathfrak{x}_{l-1}) \in \times_{j=1}^{l-1}([a_{2j}, a_{2j+1}]^+ \cup [a_{2j}, a_{2j+1}]^-)$ there is an unique $(t_1, ..., t_{l-1}) \in [-1/2, 1/2]^{l-1}$ such that

$$\mathcal{A}(\mathfrak{x}_1, ..., \mathfrak{x}_{l-1}) = (B_{ij})\left((t_1, ..., t_{l-1})^t + (m_1, ..., m_{l-1})^t\right) \tag{69}$$

where $m_\kappa \in \mathbb{Z}$. Using (23) we obtain

$$\sum_{\kappa=1}^{l-1}\left(\frac{1}{2}\sum_{j=1}^{l-1}\delta_{j,n_\nu}\omega_\kappa(x_{j,n_\nu})\right)B_{k\kappa} = \sum_{j=1}^{l-1}\frac{1}{2}\int_{\mathfrak{x}^*_{j,n_\nu}}^{\mathfrak{x}_{j,n_\nu}}\varphi_k = \sum_{\kappa=1}^{l-1}t_{\kappa,n_\nu}B_{k\kappa}$$

with $t_{\kappa,n_\nu} \in [-1/2, 1/2]$ for $k = 1, ..., l-1$, hence

$$t_{\kappa,n_\nu} = \frac{1}{2}\sum_{j=1}^{l-1}\delta_{j,n_\nu}\omega_\kappa(x_{j,n_\nu}) \text{ modulo } 1 \tag{70}$$

Since, by (69), $(\mathfrak{x}_{1,n_\nu},...,\mathfrak{x}_{l-1,n_\nu})_{\nu\in\mathbb{N}}$ converges if and only if $(t_{1,n_\nu},...,t_{l-1,n_\nu})_{\nu\in\mathbb{N}}$ converges modulo 1 the assertion follows by (14) and (70).

b) Follows immediately by Theorem 3.2 b) and Proposition 5.2 b). □

Moreover under the assumptions of Theorem 5.3 b) the number of accumulation points of the recurrence coefficients is infinite, which has been proved for the isospectral torus in [12] already, see also [11].

Corollary 5.4. *Let (n_ν) be such that $(\boldsymbol{\alpha}^{l-1}_{1+n_\nu}(d\mu),\boldsymbol{\lambda}^{l-1}_{2+n_\nu}(d\mu))_{\nu\in\mathbb{N}}$ converges. Then the sequence $(n_{\nu+1}-n_\nu)_{\nu\in\mathbb{N}}$ is unbounded if $1,\omega_1(\infty),...,\omega_{l-1}(\infty)$ are linearly independent over $\mathbb{Q}$.*

Proof. By (12) and Theorem 5.3 a) we get that the RHS of the system of equations $k=1,...,l-1$,

$$\begin{aligned}&(n_{\nu+1}-n_\nu)\omega_k(\infty)-2(m_{k,n_{\nu+1}}-m_{k,n_\nu})=\\ =&\frac{1}{2}\sum_{j=1}^{l-1}(\delta_{j,n_{\nu+1}}\omega_k(x_{j,n_{\nu+1}})-\delta_{j,n_\nu}\omega_k(x_{j,n_\nu}))\end{aligned}\tag{71}$$

tends to zero, where $m_k,n_{\nu+1},m_k,n_\nu\in\mathbb{Z}$. Assume that $(n_{\nu+1}-n_\nu)$ is bounded and thus that the LHS takes on modulo 1 a finite number of values only and is not equal to zero, since $\omega_k(\infty)$ must not be rational, which yields the desired contraction. □

In particular Theorem 5.3 holds true for measures from the isospectral torus and thus holds true even for measures, whose recurrence coefficients satisfy (10), which are described in [20]. Therefore it looks like we could have restricted to the isospectral torus. The problem is that for accumulation points of zeros which is the other main point of our studies there are no corresponding statements like (10) available.

Thus the sequences (n_ν) for which the recurrence coefficients $(\boldsymbol{\alpha}^{l-1}_{1+n_\nu}(d\mu)$, $\boldsymbol{\lambda}^{l-1}_{2+n_\nu}(d\mu))_{\nu\in\mathbb{N}}$ converge are determined by the harmonic measures E, only the values of the accumulation points depend on μ. We believe that the property "$(\boldsymbol{\alpha}^{l-1}_{1+n_\nu}(d\sigma)$, $\boldsymbol{\lambda}^{l-1}_{2+n_\nu}(d\sigma))_{\nu\in\mathbb{N}}$ converges if and only if $(n_\nu\boldsymbol{\omega}(\infty))_{\nu\in\mathbb{N}}$ converges modulo 1" holds for a wide class of measures σ whose essential support is E; loosely speaking that it is the counterpart of the measures whose essential support is a single interval and for which the recurrence coefficients converge, for instance as in Rakhmanov's [24] and Denisov's [4] case.

As an immediate consequence of Thm. 5.3 and Prop. 5.2 in conjunction with Cor. 2.5 we obtain

Corollary 5.5. *The Green's functions $(G(z,n_\nu,n_\nu))_{\nu\in\mathbb{N}}$ and $(G(z,1+n_\nu,$ $n_\nu))_{\nu\in\mathbb{N}}$, defined in (18) and (19), converge simultaneously uniformly on compact subsets of Ω if and only if $(n_\nu\boldsymbol{\omega}(\infty))_{\nu\in\mathbb{N}}$ converges modulo 1.*

Theorem 5.6. *The following statements hold for the recurrence coefficients of any measure μ of the form (8) with $w \in Sz(E)$. a) For every $k \in \mathbb{Z}$ $\lim_\nu(\boldsymbol{\alpha}^{l-1}_{k+1+n_\nu}, \boldsymbol{\lambda}^{l-1}_{k+2+n_\nu})$ exists if $\lim_\nu(\boldsymbol{\alpha}^{l-1}_{1+n_\nu}, \boldsymbol{\lambda}^{l-1}_{2+n_\nu})$ exists.*

b)Put $(\tilde{\boldsymbol{\alpha}}^{l-1}_{k+1+(n_\nu)}, \tilde{\boldsymbol{\lambda}}^{l-1}_{k+2+(n_\nu)}) := \lim_\nu(\boldsymbol{\alpha}^{l-1}_{k+1+n_\nu}, \boldsymbol{\lambda}^{l-1}_{k+2+n_\nu})$. The limits are related to each other by

$$(\tilde{\boldsymbol{\alpha}}^{l-1}_{k+1+(n_\nu)}, \tilde{\boldsymbol{\lambda}}^{l-1}_{k+2+(n_\nu)}) = \psi^k((\tilde{\boldsymbol{\alpha}}^{l-1}_{1+(n_\nu)}, \tilde{\boldsymbol{\lambda}}^{l-1}_{2+(n_\nu)}))$$

where ψ is a continuous map on $\mathbb{R}^{l-1} \times \mathbb{R}^{l-1}_+$ which does not depend on μ and ψ^k denotes the k-th composition. (Note that $\eta^{-1} \circ \psi^k \circ \eta$, where $\eta = \mathcal{I}^{-1}_{l-1} \circ \tau$, maps the corresponding points on the torus to each other).
c) The orbit $\{\psi^k((\tilde{\boldsymbol{\alpha}}^{l-1}_{1+(n_\nu)}, \tilde{\boldsymbol{\lambda}}^{l-1}_{2+(n_\nu)})) : k \in \mathbb{N}_0\}$ is dense in the set of accumulation points of the sequence $(\boldsymbol{\alpha}^{l-1}_{1+n}, \boldsymbol{\lambda}^{l-1}_{2+n})_{n\in\mathbb{N}}$ if $1, \omega_1(\infty), ..., \omega_{l-1}(\infty)$ are linearly independent over $\mathbb{Q}$.

Proof. Part a) follows by Proposition 5.2 b) and the fact that by (40) and (41) the limits $\lim_\nu \int x^j P^2_{n_\nu}$ and $\lim_\nu \sqrt{\lambda_{2+n_\nu}} \int x^j P_{n_\nu} P_{1+n_\nu}$ exist for every $j \in \mathbb{N}$ if they exist for $j = 0, ..., l-1$.

b) Let us first show the assertion for $k = 1$. Taking a look at $(\tilde{\boldsymbol{\alpha}}^{l-1}_{2+(n_\nu)}, \tilde{\boldsymbol{\lambda}}^{l-1}_{3+(n_\nu)})$ we see that we have to show only that

$$\begin{aligned} &\lim_\nu \alpha_{[\frac{l-1}{2}]+2+n_\nu} = f(\tilde{\boldsymbol{\alpha}}^{l-1}_{1+(n_\nu)}, \tilde{\boldsymbol{\lambda}}^{l-1}_{2+(n_\nu)}) \text{ and} \\ &\lim_\nu \lambda_{[\frac{l-1}{2}]+3+n_\nu} = g(\tilde{\boldsymbol{\alpha}}^{l-1}_{1+(n_\nu)}, \tilde{\boldsymbol{\lambda}}^{l-1}_{2+(n_\nu)}), \end{aligned} \tag{72}$$

where f, g are continuous functions. By (41) and Proposition 5.2 b)

$$\lim_\nu \sqrt{\lambda_{2+n_\nu}} \int x^l P_{1+n_\nu} P_{n_\nu} = h_1(\tilde{\boldsymbol{\alpha}}^{l-1}_{1+(n_\nu)}, \tilde{\boldsymbol{\lambda}}^{l-1}_{2+(n_\nu)})$$

and thus by (63), recall (67),

$$\begin{aligned} &\lim_\nu \alpha_{\frac{l}{2}+1+n_\nu} = f_1(\tilde{\boldsymbol{\alpha}}^{l-1}_{1+(n_\nu)}, \tilde{\boldsymbol{\lambda}}^{l-1}_{2+(n_\nu)}), \text{ respectively,} \\ &\lim_\nu \lambda_{[\frac{l-1}{2}]+2+n_\nu} = g_1(\tilde{\boldsymbol{\alpha}}^{l-1}_{1+(n_\nu)}, \tilde{\boldsymbol{\lambda}}^{l-1}_{2+(n_\nu)}), \end{aligned} \tag{73}$$

if l is even, respectively, l is odd.

Next let us observe that by (60) and (67) for $n \geq l$

$$\int x^l P^2_{1+n} = \begin{cases} \lambda_{\frac{l}{2}+2+n} \prod_{i=n+3}^{n+1+\frac{l}{2}} \lambda_i + \tilde{e}_1(\alpha_{\frac{l}{2}+1+n}, \boldsymbol{\alpha}^{l-1}_{1+n}, \boldsymbol{\lambda}^{l-1}_{2+n}) & \text{if } l \text{ is even} \\ \alpha_{(\frac{l-1}{2})+2+n} \prod_{i=n+3}^{n+2+(\frac{l-1}{2})} \lambda_i + \tilde{e}_2(\lambda_{\frac{l-1}{2}+2+n}, \boldsymbol{\alpha}^{l-1}_{1+n}, \boldsymbol{\lambda}^{l-1}_{2+n}) & \text{if } l \text{ is odd,} \end{cases} \tag{74}$$

where $\tilde{e}_1, \tilde{e}_2$ are continuous functions. By (41), (74) applied to $l-1$, and Proposition 5.2 b) it follows that

$$\lim_{\nu} \int x^l P_{1+n_\nu}^2 = \begin{cases} h_2(\tilde{\alpha}_{\frac{l-2}{2}+2+(n_\nu)}, \tilde{\boldsymbol{\alpha}}_{1+(n_\nu)}^{l-1}, \tilde{\boldsymbol{\lambda}}_{2+(n_\nu)}^{l-1}) & \text{if } l \text{ is even} \\ h_3(\tilde{\lambda}_{\frac{l-1}{2}+2+(n_\nu)}, \tilde{\boldsymbol{\alpha}}_{1+(n_\nu)}^{l-1}, \tilde{\boldsymbol{\lambda}}_{2+(n_\nu)}^{l-1}) & \text{if } l \text{ is odd} \end{cases}$$

Thus by (74) and (73) we obtain that

$$\begin{aligned} &\lim_{\nu} \lambda_{\frac{l}{2}+2+n_\nu} = g_2(\tilde{\boldsymbol{\alpha}}_{1+(n_\nu)}^{l-1}, \tilde{\boldsymbol{\lambda}}_{2+(n_\nu)}^{l-1}), \text{ respectively,} \\ &\lim_{\nu} \alpha_{\frac{l-1}{2}+2+n_\nu} = f_2(\tilde{\boldsymbol{\alpha}}_{1+(n_\nu)}^{l-1}, \tilde{\boldsymbol{\lambda}}_{2+(n_\nu)}^{l-1}) \end{aligned}$$

if l is even, respectively, l is odd, and the relations (72) are proved.

For arbitrary k the statement follows immediately by iteration using induction arguments and the fact that the relations (72) hold for any (n_ν) for which we have convergence.

c) Since $(n_\nu \boldsymbol{\omega}(\infty) - \boldsymbol{c}) \underset{\nu\to\infty}{\longrightarrow} \boldsymbol{\gamma}$ modulo 1 implies that $(n_\nu + k)\boldsymbol{\omega}(\infty) - \boldsymbol{c} \underset{\nu\to\infty}{\longrightarrow} \boldsymbol{\gamma} + k\boldsymbol{\omega}(\infty)$ modulo 1 the assertion follows by part b) and Thm. 5.3 using the fact that the RHS is dense in $[0,1]^{l-1}$ with respect to k, $k \in \mathbb{Z}$. $\square$

The map ψ can be obtained explicitly by Corollary 2.5 and (67), at least for small l. Most likely Theorem 5.6 c) holds true without the assumption of linear independence of $1, \omega_1(\infty), ..., \omega_{l-1}(\infty)$. Note that Theorem 5.6 is of so-called "oracle type", see [25]. Finally let us remark that we expect (investigations are on the way) that Theorem 4.3 and Theorem 5.3 hold true for so-called homogeneous sets E and a possible infinite set of mass points lying outside E and accumulating on E.

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